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Abstract. The differential equation for the harmonic oscillator is generalised to include an interaction potential containing a positive definite quadratic denominator. Conditions are developed under which certain eigenfunctions take the form of an exponential function multiplied by a polynomial. The problem reduces to that of finding the eigenvalues of a certain matrix that is tridiagonal in form. Properties of the eigenvalues of this matrix are investigated, since they are functions of a parameter occurring in the positive definite quadratic form. The asymptotic forms of these eigenvalues are developed together with computed results expressed as curves showing the variation of the eigenvalues with respect to this parameter.

1. Motivation and introduction

Interest has centred recently on an eigenvalue problem associated with an ordinary differential equation containing two parameters instead of the usual one eigenvalue parameter; the equation has made its appearance, for example, in laser theory. Several lines of approach have been adopted in the investigation of the eigenvalues and eigenfunctions of the differential equation

$$\frac{d^2w}{dz^2}+\left(E-z^2-\frac{\lambda z^2}{1+gz^2}\right)w=0 \tag{1}$$

where z is real, g being real and non-negative, and where $|w|\to 0$ as $z\to\pm\infty$ along the real axis. E is the eigenvalue parameter, a spectrum of values existing for each real value of λ. In this paper, however, particular eigenfunctions are sought such that λ is not independent of E, these eigenfunctions existing only for particular triplets (E, λ, g). The ratio appearing in the bracket may be regarded as a perturbation term, though we have no control over the magnitude of λ in the following investigation.

Kaushal (1979) has evaluated the asymptotic expansion of E in terms of the parameter $h\equiv g/2(1+\lambda)^{1/2}$, giving the terms of the expansion up to $O(h^3)$. Numerical values of the first four eigenvalues are given for 50 pairs (g, λ), when $g=0.1$, 0.2, 0.5, 0.8, 1.0 and $\lambda=0.1$, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100. Mitra (1978) has considered the equation for $g, \lambda>0$, and has calculated the first three eigenvalues for 90 pairs (g, λ), with $g=0.1$, 0.5, 1, 2, 5, 10, 20, 50, 100 and $\lambda=0.1$, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100. Biswas *et al* (1973), using the Hill determinant method, calculated the eigenvalues of the equation

$$d^2w/dz^2+(E-z^2-\lambda z^{2m})w=0$$

for $m = 2, 3, 4$ and $0.1 \leq \lambda \leq 100$, with a final remark that their method can also be used for the $\lambda z^2/(1+gz^2)$ interaction. Bessis and Bessis (1980) have given extensive numerical calculations based on the variational method with the harmonic oscillator functions as the basis set. Their principal table of results gives the first four eigenvalues for 121 pairs (g, λ), with $g, \lambda = 0.1, 0.5, 1, 2, 5, 10, 20, 50, 100, 200, 500$. Flessas (1981) has given a brief investigation when $\lambda < 0$, in which eigenfunctions have been found in the form $e^{-z^2/2}u(z)$, where u is a polynomial in z^2, of degree 2 or 4 in z. His treatment does not show the nature of the generalisation to polynomials of arbitrary degree, nor the properties enjoyed by the eigenvalues and eigenfunctions in the general case.

In the present paper, we generalise the investigation of Flessas. By adopting a more systematic approach, and by observing the existence of a necessary factor in the polynomial u, we produce in matrix form the equations yielding the coefficients appearing in the polynomial of general degree. The parameters E and λ in equation (1) are derived from the characteristic roots of an infinite matrix that is truncated without approximation; its characteristic vectors provide the coefficients of the polynomial. Properties of the roots are examined, and their behaviour as functions of g is discussed and illustrated numerically, attention being paid to their asymptotic forms. The relationships of the eigenvalues and eigenfunctions of equation (1) to those associated with non-perturbed polynomial-type forms are derived.

2. The series solution

Rearrangement of the expression in brackets in equation (1) yields

$$\frac{d^2w}{dz^2}+\left(G-z^2+\frac{\mu}{1+gz^2}\right)w=0 \tag{2}$$

where $E = G+\lambda/g$, $\lambda = \mu g$. Since the asymptotic form of the required eigenfunction will contain the factor $e^{-z^2/2}$, we introduce the change of dependent variable $w = e^{-z^2/2}u$, yielding

$$\frac{d^2u}{dz^2}-2z\frac{du}{dz}+\left(H+\frac{\mu}{1+gz^2}\right)u=0 \tag{3}$$

where $G = H+1$, $E = H+1+\lambda/g$.

If a polynomial solution exists for a particular pair of values of H and μ, it *must* contain the factor $1+gz^2$, in order to cancel the identical factor in the denominator of the coefficient of u. This simplifying feature has not, to our knowledge, been noted before, and certainly it is not obvious that this is a factor of the polynomial of degree 4 explicitly calculated by Flessas (1981). Write, therefore, $u = (1+gz^2)p$, where p is a power series generally, or in our present investigation a polynomial whose degree is two less than that of u.

Substitution into equation (3) gives

$$(1+gz^2)p''+4gzp'+2gp-2z(1+gz^2)p'-4gz^2p+H(1+gz^2)p+\mu p=0. \tag{4}$$

Clearly p contains powers of z that ascend in steps of 2.

If p commences with the term z^c, the indicial equation arises only from the single term p'', namely $c(c-1)=0$, with roots 0, 1. We shall restrict ourselves only to the former value $c=0$; the latter value is treated by the same method. (Of course no logarithmic singularity can arise, since the equation has no singularity at $z = 0$.) Flessas failed to mention this second case, owing to his substitution $z^2 = t$ apparently suggesting that polynomials had to commence with a constant, whereas a terminating series commencing with $t^{1/2}$ would also yield a polynomial in z. Two such polynomials are not simultaneously solutions of a particular equation (4), since different values of H and μ are involved in each case.

If it is possible for u to be a polynomial of degree $2n$ in z, then p will be of degree $2n-2$. Write

$$p_{2n-2} = \sum_{j=0}^{n=1} a_j z^{2j}, \qquad a_0 = 1, \qquad a_{n-1} \neq 0$$

and we seek conditions for such polynomials to exist.

In equation (4), denote by $-q$ the constant coefficient of p

$$-q = H + 2g + \mu \tag{5}$$

so

$$E = -q - 2g + 1. \tag{6}$$

When a polynomial solution is possible, the substitution of p_{2n-2} into equation (4) yields $2n$ as the highest power of z. Its coefficient, which must vanish, is

$$[-2g(2n-2) - 4g + Hg]a_{n-1}$$

yielding $H = 4n$. Hence, from (5)

$$\lambda = \mu g = -g(q + 4n + 2g). \tag{7}$$

The permanent relation between E and λ is, from (6) and (7),

$$E - \lambda/g = 4n + 1. \tag{8}$$

Since this value equals G, equation (2) becomes

$$\frac{d^2 w}{dz^2} + \left(4n + 1 - z^2 + \frac{\mu}{1 + gz^2}\right) w = 0. \tag{9}$$

We shall investigate later the orthogonality properties of this equation.

The complete substitution of p_{2n-2} into equation (4) with $H = 4n$ yields the coefficients of $1, z^2, \ldots, z^{2n}$. These must vanish, and written in matrix form these are

$$\begin{pmatrix} -q & 2.1 & 0 & \cdots & \cdots \\ 4(n-1)g & \begin{matrix} 2.5g \\ -2.2-q \end{matrix} & 4.3 & \cdots & \cdots \\ 0 & 4(n-2)g & \begin{matrix} 4.7g \\ -2.4-q \end{matrix} & \cdots & \cdots \\ 0 & 0 & 4(n-3)g & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \begin{matrix} (2n-2)(2n+1)g \\ -2(2n-2)-q \end{matrix} \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ \cdots \\ a_{n-1} \end{pmatrix} = 0. \tag{10}$$

The $n \times n$ matrix in this equation is tridiagonal in form.

Obviously the q are the characteristic roots of, and the coefficients of the polynomials form the corresponding characteristic vectors of, the $n \times n$ matrix whose elements are given by

$$\begin{array}{ll} \text{the diagonal elements} & 2(i-1)[(2i+1)g-2] \equiv X_i g + Y_i \\ \text{the above-diagonal elements} & 2i(2i-1) \equiv Y_{i,i+1} \\ \text{the below-diagonal elements} & 4(n-i+1)g \equiv X_{i,i-1} g. \end{array} \tag{11}$$

The elements are linear in the parameter g.

Hence when $n=1$, the one value of q obviously vanishes, and $E=1-2g$, $\lambda=-g(2g+4)$, where $\lambda<0$ when $g>0$.

When $n=2$, matrix (11) is

$$\begin{pmatrix} 0 & 2 \\ 4g & 10g-4 \end{pmatrix}$$

with

$$\begin{aligned} q &= 5g-2 \pm (25g^2-12g+4)^{1/2} \\ E &= -7g+3 \mp (25g^2-12g+4)^{1/2} \\ \lambda &= -7g^2-6g \mp g(25g^2-12g+4)^{1/2} \end{aligned}$$

with the two values of λ negative when $g>0$, these values having been given by Flessas. We shall later prove that λ is negative for all values of n.

3. Reduction to the Hermite polynomials

When $\lambda=0$, the equation reduces to

$$\mathrm{d}^2w/\mathrm{d}z^2+(E-z^2)w=0.$$

The assumption that $u=(1+gz^2)p_{2n-2}(z)$ is a polynomial of degree $2n$ demands that $1+gz^2$ should be a factor of the Hermite polynomial of degree $2n$. Moreover, since $\lambda=0$, we have $q=-4n-2g$, so the characteristic equation is no longer an equation for q, but an equation for g. The solutions $g_1, g_2, \ldots, g_n$ of the equation

$$\begin{vmatrix} 2g+4n & 2.1 & 0 & \cdots \\ 4(n-1)g & 12g+4n-4 & 4.3 & \cdots \\ 0 & 4(n-2)g & 30g+4n-8 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{vmatrix} = 0$$

will yield the polynomial of degree $2n$

$$(1+g_1z^2)(1+g_2z^2)\ldots(1+g_nz^2)$$

proportional to the Hermite polynomial of degree $2n$.

Apart from a numerical factor, this product is obtained from the determinant by substituting $-1/z^2$ for g, and multiplying the determinant by z^{2n}, giving

$$H_{2n}(z) \propto \begin{vmatrix} -2+4nz^2 & 2z^2 & 0 & \cdots \\ -4(n-1) & -12+4(n-1)z^2 & 12z^2 & \cdots \\ 0 & -4(n-2) & -30+4(n-2)z^2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{vmatrix}. \tag{12}$$

The coefficient of z^{2n} in this Hermite polynomial must be 2^{2n}, but in the determinant it is $4^n n!$. We must therefore divide (12) by $n!$.

Thus, for example, when $n=3$,

$$H_6(z) = \frac{1}{3!}\begin{vmatrix} -2+12z^2 & 2z^2 & 0 \\ -8 & -12+8z^2 & 12z^2 \\ 0 & -4 & -30+4z^2 \end{vmatrix}$$

$$= 64z^6 - 480z^4 + 720z^2 - 120.$$

4. Orthogonality properties

Equation (9) is of the form

$$\mathrm{d}^2w/\mathrm{d}z^2 + [f(z) + \mu h(z)]w = 0 \tag{13}$$

where, for a given integer n (thereby fixing $f(z)$) and real g, there are n values of μ found by means of the characteristic equation of matrix (11). In order to assess the properties of the q–g curves for $g \geq 0$, it is necessary to let g be negative in equation (1). Certainly on physical grounds $g \geq 0$, but on mathematical grounds nothing that has been discussed so far prevents us from taking $g < 0$. Equation (1) then has a singularity on the real z axis when $z^2 = -1/g$, but the particular eigensolutions under discussion have no singularity at such points, although the second independent solution (not considered here) of any equation would have a singularity since the singularity of the equation is not apparent. When such particular solutions are substituted into equation (1), the factor $1+gz^2$, $g<0$, cancels, leaving no singularity, so the equation or its equivalent form may be integrated along the real axis if necessary.

If some μ values, related directly to the solutions of an nth degree polynomial equation with real coefficients, are complex they will occur in conjugate pairs μ and μ^* say. When μ^* appears in equation (13), its solution will be w^*. In the usual way, the multiplication of (13) by an eigenfunction w^* and its conjugate equation by w, and the subtraction of these results, yields

$$0 = (\mu^* - \mu)\int_{-\infty}^{\infty} \mathrm{e}^{-z^2}(1+gz^2)pp^*\,\mathrm{d}z. \tag{14}$$

Provided $g \geq 0$, the integrand is positive for all z, so $\mu = \mu^*$ and is therefore real; q is therefore real from (7). Now any branch q (and hence μ) is a continuous function of g, being a root of a polynomial equation whose coefficients are continuous functions of g. Moreover, the integrand of (14) consists of $2n$ integrals of the form

$$\int_{-\infty}^{\infty} \mathrm{e}^{-z^2} z^{2r}\,\mathrm{d}z \qquad r = 0, 1, 2, \ldots, 2n-1$$

whose coefficients are functions of g through the coefficients appearing in the polynomial p. Consequently, the integral in (14) cannot vanish unless g has any real value that causes this sum of $2n$ integrals to vanish (the sum cannot be identically zero, since it cannot vanish for $g \geqslant 0$). Whether such values of g actually exist is irrelevant in our argument; it is not necessary to investigate the possibility. Hence, apart from these possible values of g, $\mu^* - \mu = 0$ and μ is real. But μ is a continuous function of g, so when g has one of these special values, μ must still be real. The conclusion is that the values of μ (and hence of q) are always real for real g.

Now consider two distinct values of μ: μ_1 and μ_2 for given n and g. The same treatment as before yields

$$0 = (\mu_1 - \mu_2) \int_{-\infty}^{\infty} e^{-z^2}(1+gz^2)p_1p_2 \, dz$$

implying that this integral vanishes.

Using the value of μ given by (7), the differential equation (9) becomes

$$\frac{d^2w}{dz^2} + \left(4n+1-z^2-\frac{q+4n+2g}{1+gz^2}\right)w = 0 \tag{15}$$

where, when n is given, q and g are related by the characteristic equation. Suppose now that n and a definite real value of q are given; the vanishing of the determinant of the matrix appearing in (10) yields a polynomial equation of degree $n-1$ in g with real coefficients which are polynomials in q. Its solutions for g are either real, or else they occur in complex conjugate pairs, in which case let g and g^* denote two roots.

This equation for g has $n-1$ roots provided the coefficient of g^{n-1} does not vanish. By expansion down the first column of the determinant, this coefficient is found to be

$$-[10q+8(n-1)](4.7)(6.9)\ldots(2n-2)(2n+1);$$

this vanishes when $q = -\frac{4}{5}(n-1)$. Provided the coefficient of g^{n-2} does not vanish at this particular value, the equation now has $n-2$ finite roots for g, the remaining root being infinite. The coefficient of g^{n-2} may be found by a method similar to that given in § 6; when $q = \frac{4}{5}(n-1)$, it is easily shown that this coefficient is proportional to $(7q+12n+4)$, which cannot vanish at this special value of q when n is a positive integer. This implies that $n-2$ finite roots definitely exist at this value of q.

The coefficients of the polynomial p_{2n-2} are functions of n, q and g. Hence we note that $w^* = e^{-z^2/2}(1+g^*z^2)p^*_{2n-2}$ satisfies the equation conjugate to (15). Then, as before,

$$\int_{-\infty}^{\infty}\left(\frac{q+4n+2g}{1+gz^2} - \frac{1+4n+2g^*}{1+g^*z^2}\right)e^{-z^2}(1+gz^2)(1+g^*z^2)p_{2n-2}p^*_{2n-2}\, dz = 0$$

or

$$(g^*-g)\int_{-\infty}^{\infty}[(q+4n)z^2-2]\,e^{-z^2}p_{2n-2}p^*_{2n-2}\, dz = 0.$$

If $q \leqslant -4n$, the integrand is negative for all z, hence g must be real. But if $q > -4n$, such a conclusion cannot be drawn immediately. However, the previous argument may again be used, to show that g is always real.

These facts will be used in the next section to describe the q–g curves and their properties.

5. The *q–g* curves

The characteristic equation gives the relation between q and g. Regarded as an equation of degree n in q, its coefficients are continuous polynomials in g, so the roots are also continuous functions of g. Moreover, q cannot become infinite for finite values of g. We have shown that for each real value of g there are n real values of q. Conversely, regarded as an equation of degree $n-1$ in g, its roots are continuous functions of q except when the coefficient of g^{n-1} vanishes. We have already calculated this value to be $q=-\frac{4}{5}(n-1)$, so the graph has an asymptote at this value of q. Apart from this asymptote, we have shown that for each real value of q there are $n-1$ real values of g.

Computer calculations of the characteristic roots q for various positive and negative values of g enable us to present figures 1–3. These depict the q–g curves for $n=3$ (3 branches of q) and $n=10$ (10 branches of q). 2 and 9 branches respectively extend to infinity in the first quadrant, 3 and 10 in the third quadrant, and one in the fourth quadrant. The horizontal asymptotes are given by $q=-\frac{4}{5}$ and $-\frac{36}{5}$ respectively. There are 2 and 9 non-horizontal asymptotes, and the curves quickly approach these even for moderate values of $|g|$; the equations of these asymptotes are derived in the next section.

When $n=10$, figure 3 magnifies these curves near the origin. When $g=0$, the values of q are obviously $0, -4, -8$ and $0, -4, \ldots, -36$ in the two cases. The curves are squeezed into the very narrow gap formed by the two outermost curves with the horizontal asymptotes.

For a general value of n, the n curves sweep monotonically and smoothly from the bottom left to the top right, with $\mathrm{d}q/\mathrm{d}g>0$. It is impossible for any curve to have

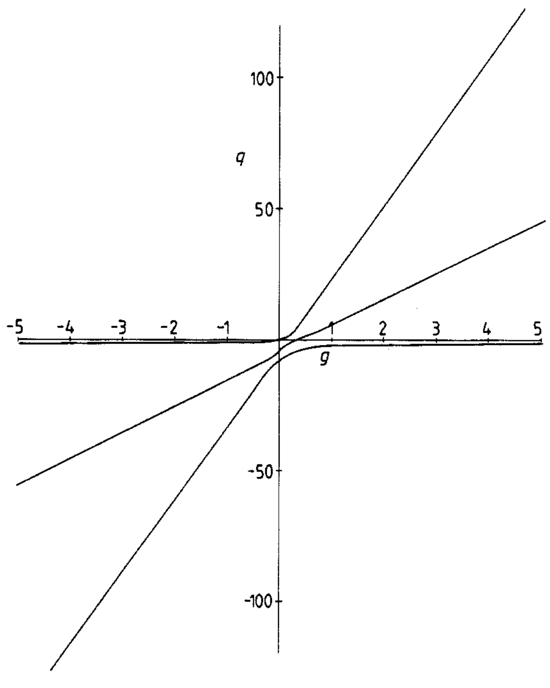

Figure 1. The three q–g curves when $n=3$, $-5\leqslant g\leqslant 5$.

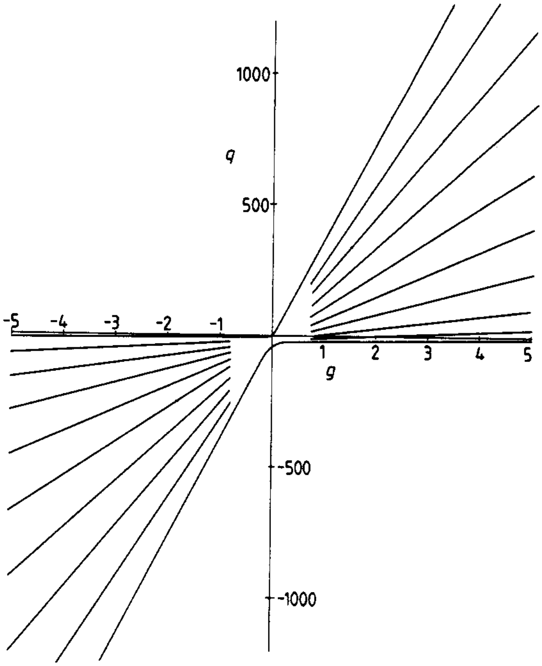

Figure 2. The ten q–g curves when $n = 10$, $-5 \leqslant g \leqslant 5$.

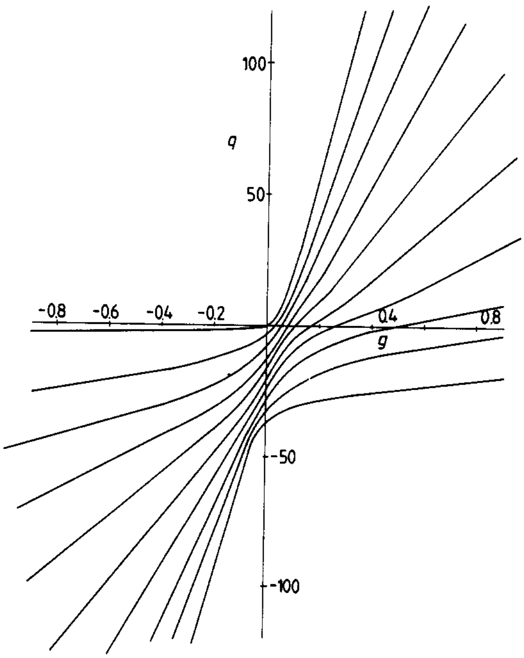

Figure 3. Magnified q–g curves when $n = 10$, $-0.8 \leqslant g \leqslant 0.8$.

a maximum, and so bend downwards with increasing g. For if a curve has a maximum at the point (g_0, q_0), then the line $q = q_0$ would cut the curves in at least n points (two coinciding at the maximum), which is impossible, since there are only $n-1$ such intersections.

This argument shows that, for all positive values of g (the physical case), every value of q is greater than the least value of q occurring when $g = 0$, namely

$$q > -4(n-1).$$

From (7)

$$-\lambda/g = q + 4n + 2g > -4(n-1) + 4n + 2g = 2g + 4$$

showing that

$$\lambda > -(2g+4)g$$

namely, λ is always negative for positive g. We have seen that $\lambda = -g(2g+4)$ when $n = 1$; the inequality does not apply in this case since the one q–g curve is merely the horizontal straight line $q = 0$.

6. The asymptotic lines of the q–g curves

To find the asymptotes of the q–g curves, we note that the characteristic equation of matrix (11) is an algebraic equation in the variables q and g, enabling us to employ the standard method for this case (see Sneddon 1976, p 38). Substitute $q = Ag + B$, equating the coefficients of g^n and g^{n-1} to zero. If solutions for A and B exist, this method not only provides their values but also proves the existence of the asymptotic lines.

Let C_n denote the determinant of the square matrix in (10), and let C_r denote the determinant formed by the first r rows and r columns. Expanding along the bottom row, we have

$$C_n = (X_n g + Y_n - q)C_{n-1} - 2(n-1)(2n-3)\cdot 4gC_{n-2}.$$

We use the abbreviation

$$Z_i = 2(i-1)(2i-3)\cdot 4(n-i+1)$$

so

$$C_n = (X_n g + Y_n - q)C_{n-1} - Z_n g C_{n-2}$$

and similarly

$$C_{n-1} = (X_{n-1} g + Y_{n-1} - q)C_{n-2} - Z_{n-1} g C_{n-3}$$
$$\vdots$$
$$C_2 = (X_2 g + Y_2 - q)C_1 - Z_2 g C_0$$

where $C_1 = -q$, $C_0 = 1$. Here, we use the symbols introduced in (11).

When we substitute $q = Ag + B$, C_r is a polynomial of degree r in q, so we write

$$C_r = E_r g^r + F_r g^{r-1} + \ldots$$

where E_r and F_r are independent of g. Hence, including all terms g^n and g^{n-1}, we have

$$C_n = [(X_n - A)g + Y_n - B](E_{n-1}g^{n-1} + F_{n-1}g^{n-2}) - Z_n g(E_{n-2}g^{n-1}) + \ldots$$

and similarly for all the other equations for the C. Now C_n vanishes, and equating coefficients of g^n and g^{n-1} (and other powers respectively in the other equations), we obtain the series of equations

$$\begin{aligned}
&0=(X_n-A)E_{n-1} && 0=(X_n-A)F_{n-1}+(Y_n-B)E_{n-1}-Z_nE_{n-2}\\
&E_{n-1}=(X_{n-1}-A)E_{n-2} && F_{n-1}=(X_{n-1}-A)F_{n-2}+(Y_{n-1}-B)E_{n-2}-Z_{n-1}E_{n-3}\\
&\quad\vdots && \quad\vdots\\
&E_2=(X_2-A)(-A) && F_2=(X_2-A)(-B)+(Y_2-B)(-A)-Z_2.
\end{aligned}$$

It follows that

$$A=X_{n-k} \qquad k=0,1,\ldots,n-1$$

giving the n gradients of the n asymptotic lines, including the horizontal line $k=n-1$. Then

$$E_{n-1}=E_{n-2}=\ldots=E_{n-k}=0$$

and

$$E_{n-k-1}=(X_{n-k-1}-X_{n-k})(X_{n-k-2}-X_{n-k})\ldots(X_2-X_{n-k})(-X_{n-k})$$

$$E_{n-k-2}=(X_{n-k-2}-X_{n-k})\ldots(X_2-X_{n-k})(-X_{n-k})$$

and so on.

As far as the equations for the F are concerned, $F_{n-1}=0$ if E_{n-1} and E_{n-2} vanish. The first equation with non-vanishing terms is

$$0=F_{n-k+1}=(X_{n-k+1}-X_n)F_{n-k}+(Y_{n-k+1}-B)\cdot 0-Z_{n-k+1}E_{n-k+1}$$

and

$$F_{n-k}=0\cdot F_{n-k-1}+(Y_{n-k}-B)E_{n-k-1}-Z_{n-k}E_{n-k-2}.$$

Eliminating F_{n-k}, and using the values of the E, we find that

$$B=Y_{n-k}-\frac{Z_{n-k}}{X_{n-k-1}-X_{n-k}}-\frac{Z_{n-k+1}}{X_{n-k+1}-X_n}.$$

Using the values of X, Y, Z, and replacing $n-k$ by j, $j=1,2,\ldots,n$, we obtain

$$A=2(j-1)(2j+1)$$

$$B=-4(j-1)+\frac{4(j-1)(2j-3)(n-j+1)}{4j-3}-\frac{4j(2j-1)(n-j)}{4j+1}$$

yielding the equations of the n asymptotes. For example, when $n=3$,

$$q=-\tfrac{8}{5} \qquad q=10g-\tfrac{76}{15} \qquad q=28g-\tfrac{16}{3}$$

straight lines that are clearly discerned in figure 1.

The asymptote with zero gradient is given when $j=1$, namely $q=\frac{4}{5}(n-1)$, a value already derived. The steepest gradient occurs when $j=n$, the line being

$$q=2(n-1)(2n+1)g-\frac{8n(n-1)}{4n-3}. \tag{16}$$

Since A is independent of n (except that j extends up to n), some gradients of asymptotes are identical for distinct values of n.

From the particular case $n = 2$, Flessas surmised the hypothesis that as the degree of the polynomial p_{2n-2} increases, with the value of g maintained in the range $0 < g \leq 1$, then large positive values of E can be obtained. Our results prove this statement. For the lowest q–g curve is such that $q \doteqdot -4(n-1)$ for small g, and so from (6)

$$E = -q - 2g + 1 \doteqdot 4n - 2g - 3$$

increasing indefinitely with n. This result is also true for larger values of g when the asymptotic value of the lowest q is used

$$E \sim \tfrac{4}{5}(n-1) - 2g + 1$$

though n must be much larger than g so that E does not become negative. But the result is not true for those q–g curves that have a higher position. Using the asymptotic values for the highest curve (16), we have

$$E \sim -2(n-1)(2n+1)g + \frac{8n(n-1)}{4n-3} - 2g + 1$$

so for large n, we have $E \sim -4n^2 g$, opposite in sign to that proposed in the hypothesis.

To notice how the exact and asymptotic values of q compare for $n = 10$, table 1 gives the values of q when $j = 2$ and 10, as given by computer calculation and by the two straight line asymptotes with equations

$$q = 10g - \tfrac{272}{15} \qquad j = 2 \qquad q = 378g - \tfrac{720}{37} \qquad j = 10.$$

When $j = 10$, it is noteworthy how accurate the values of q are, even when $g = 1$.

Table 1.

	$j=2, n=10$		$j=10, n=10$	
g	q exact	q asymptotic	q exact	q asymptotic
1	−2.462234	−8.133333	358.802934	358.540540
2	5.053647	1.866667	736.671728	736.540540
3	13.670505	11.866667	1114.627997	1114.540540
4	23.040458	21.866667	1492.606132	1492.540540
5	32.712668	31.866667	1870.593013	1870.540540

7. Theorem relating to the eigenfunctions

The characteristic vectors of matrix (11) provide the n coefficients appearing in the polynomial p_{2n-2} for each of the n values of q. We shall denote the n values of q by $q_0^{(n)}, q_1^{(n)}, \ldots q_{n-1}^{(n)}$ in descending order. In figure 4, we have calculated the corresponding eigenfunctions $w_0^{(n)}, w_1^{(n)}, w_2^{(n)}$ when $n = 3$ for $g = 0, 1, 2$, the first value of g corresponding to the Hermite polynomials H_0, H_2, H_4. Since these polynomials are all even, we use the z^2 scale, with $z^2 \geq 0$ only. The curves are not normalised in any sense; the vertical scale is therefore without significance.

For these chosen values of g, it can be seen that $w_0^{(3)}$ has no real zeros; $w_1^{(3)}$ has one real zero, and $w_2^{(3)}$ has two real zeros for $z \geq 0$, and similarly for $z \leq 0$. This feature has a deeper significance for all values of n and all positive values of g.

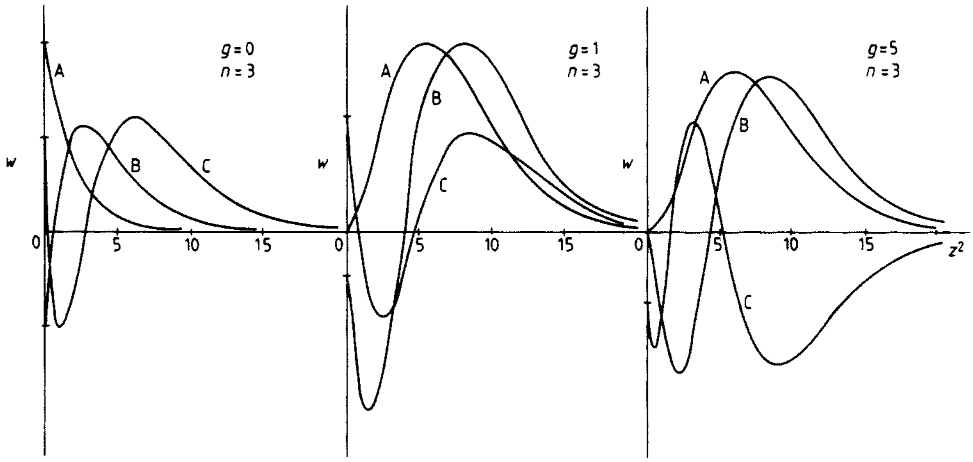

Figure 4. The three eigenfunctions w against z^2 when $n = 3$ for $g = 0, 1, 5$; A corresponds to the largest value of q ($w(z^2)$ has no zeros), B to the middle value of q ($w(z^2)$ has one zero), C to the lowest value of q ($w(z^2)$ has two zeros).

We rewrite equation (15) as

$$\frac{d^2w}{dz^2}+\left[\left(4n+1-z^2-\frac{4n+2g}{1+gz^2}\right)-\frac{q}{1+gz^2}\right]w=0 \tag{17}$$

and consider this for fixed n and $g>0$. We have seen that there are n values of q yielding n polynomials of degree $2n$. There will be at most $2n-2$ real zeros of this polynomial, since the implicit factor $1+gz^2$ cannot contribute real zeros.

Quite apart from these n polynomials, for given n and g equation (17) and the condition $w\to 0$ as $z\to\pm\infty$ form a Sturm–Liouville system, with q as the eigenparameter, so there exists a decreasing sequence of eigenvalues of q: $Q_0, Q_1, Q_2, \ldots$, say, the corresponding eigenfunctions W_j possessing j real zeros in keeping with standard theory. When $j=2n$, the eigenfunction will have $2n$ real zeros, while for $j>2n$ the number of zeros exceeds $2n$. Consequently the values of q that yield the polynomial-type eigenfunctions must be $Q_0, Q_2, Q_4, \ldots, Q_{2n-2}$, the largest being Q_0. So Q_0 yields the polynomial of degree $2n$ with no real zeros, and generally $Q_{2r}=q_r$ yields the polynomial $(1+gz^2)p_{2n-2}$ with $2r$ real roots. That is, these n even polynomials with the exponential factor $e^{-z^2/2}$ *form the first n even eigenfunctions* of equation (17). The subsequent eigenfunctions do not contain a polynomial factor.

If, finally, we consider the odd eigenfunctions when the root $c=1$ of the indicial equation is used, we find that

$$u=(1+gz^2)z(1+a_1z^2+a_2z^4+\ldots+a_{n-1}z^{2n-2})$$

requires $H=4n+2$, and that the equation for w is

$$\frac{d^2w}{dz^2}+\left(4n+3-z^2-\frac{4n+2+2g}{1+gz^2}-\frac{q}{1+gz^2}\right)w=0. \tag{18}$$

With the same conditions as $z\to\pm\infty$, this forms a Sturm–Liouville system *distinct* from (17), though it is amenable to the same basic treatment. Hence the odd polynomial-type eigenfunctions produced for fixed n and g are *not* the ones associated with the gaps $Q_1, Q_3, \ldots$ of the spectrum yielded by (17). Equation (18) will have its own gaps in its spectrum. The gaps in both sequences of eigenfunctions must be filled by functions distinct from polynomial-types.

Equation (17) and the condition as $z \to \pm\infty$ form a Sturm–Liouville system for any value of n, but our method calculates some of the eigenvalues for the integers $n = 1, 2, \ldots$. When $g = 1$, these have been calculated for $1 \leq n \leq 10$, and the results are exhibited in figure 5. (Apart from the range $-20 < q < 20$ approximately, these curves could have been deduced from the asymptotic form of q.) The overall shape of the Q_0, $Q_2, \ldots$ curves is evident. When n is not an integer, the values of $Q_0, Q_2, \ldots$ can be read off approximately from the graph, or an interpolation procedure may be used to calculate these intermediate values of the Q from the calculated results. Similarly by considering the Q_j against j graphs for given n, the gaps in the spectrum can be found graphically or by interpolation.

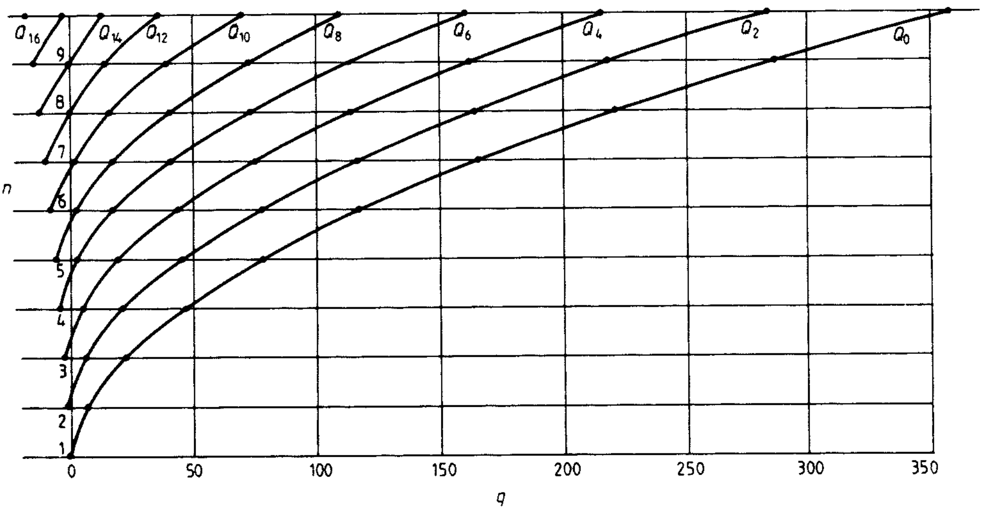

Figure 5. The n–q curves for $1 \leq n \leq 10$, showing $Q_0, Q_2, \ldots, Q_{18}$.

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Note added. Since this manuscript was accepted, a further paper by Flessas (1982) has appeared, in which definite integrals are found as solutions to equation (1).

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